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Derivation of A-stable Diagonally Implicit Superclass of Block Backward Differentiation Formula with Two Off-step Points for Solving Stiff Systems of Ordinary Differential Equations

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ABSTRACT

A new efficient hybrid block method is introduced for solving first-order stiff systems of ordinary differential equations (ODEs). The convergence conditions for the proposed implicit block numerical scheme are established, and stability analysis shows that the method is consistent, zero-stable, and convergent. The absolute stability region is plotted, confirming A-stability. The method is implemented and tested on selected stiff initial value problems, outperforming existing diagonally implicit extended 2-point superclass of block backward differentiation formula with off-step points and diagonally implicit 2-point block backward differentiation formula with two off-step points methods in terms of accuracy and computation time, especially as the step size is reduced. Hence, the results demonstrate faster convergence.

Keywords: A-stale, hybrid block method, order of the method, Stiff IVPs, ODEs

INTRODUCTION

Stiff ODEs are a class of differential equations that exhibit a unique combination of properties, making them particularly challenging to solve numerically.

These equations arise from modeling various phenomena in science and engineering, such as chemical kinetics, population dynamics, and electrical circuits. Stiff ODEs are characterized by their multi-scale nature, featuring solutions with vastly different time scales, ranging from rapid transients to slow relaxations (Suleiman *et al.* 2014).

This multi-scale behavior leads to a phenomenon known as stiffness, where the solution's sensitivity to initial conditions and parameter variations becomes extreme. As a result, traditional numerical methods can struggle to accurately integrate stiff ODEs, often leading to numerical instability, oscillations, or even failure to converge (Lambert, 1973).

In this paper, we shall be concerned with the approximate numerical integration of first-order stiff initial value problem of the form:

$$\frac{dy}{dx} = f(x,y), y(x_0) = y_0, a \le x \le b \quad (1)$$

The stiffness property demands specialized numerical techniques that can efficiently capture the diverse time scales and maintain stability. Over the years, various methods have been developed to tackle stiff ODEs, including implicit schemes such as those found in Abasi (2014), Alt (1986), Alvarez (2000), Cash (1980), Ibrahim (2007), Musa (2012), Yatim (2015), and Zawawi (2015) and so on. These methods have been refined and modified to improve their efficiency, accuracy, and robustness. This paper presents the mathematical formulation, derivation of order and error constant, stability analysis, and implementation of a two-point block hybrid method, which is a novel numerical approach for solving first-order stiff ODEs. The method is developed in terms of a free parameter ρ, which is varied within the range $\rho \epsilon (-1,1)$ to achieve optimal stability and convergence properties. By carefully selecting values of ρ that ensure stability and convergence, we demonstrate that the proposed method can produce reasonably accurate numerical approximations for stiff problems. The hybrid approach combines the advantages of different numerical methods, offering a robust and efficient tool for tackling challenging stiff ODEs

Mathematical Formulation of the Method

In this section, our focus will be on formulating the proposed two-point block hybrid method that computes two solution values with two off-step points when the step size is halved, this method is designed for the numerical integration of stiff initial value problems and has the following form:

$$\sum_{j=0}^{1} \alpha_j\, y_{n+j-1} + \sum_{j=0}^{k+1} \alpha_{j+2}\, y_{n+\frac{(j+1)}{2}} = h\beta_k \left(f_{n+k} - \rho f_{n+k-\frac{3}{2}}\right), k = \frac{1}{2}, 1, \frac{3}{2}, 2. \quad (2)$$

The method (2) is derived through Taylor series expansion about any point x_n.

First Point: $k = \frac{1}{2}$

To derive the first off-step point $y_{n+\frac{1}{2}}$, let's define the linear operator as:

$$L_{\frac{1}{2}}[y(x_n), h]: \alpha_0 y_{n-1} + \alpha_1 y_n + \alpha_2 y_{n+\frac{1}{2}} - h\beta_{\frac{1}{2}}(f_{n+\frac{1}{2}} - \rho f_{n-1}) = 0. \quad (3)$$

The associated approximate relationship for (3) can be expressed as:

$$\alpha_0 y(x_n - h) + \alpha_1 y(x_n) + \alpha_2 y(x_n + \tfrac{1}{2}h) - h\beta_{\frac{1}{2}}(f\left(x_n + \tfrac{1}{2}h\right) - \rho f(x_n - h)) = 0. \quad (4)$$

Expanding equation (4) as a Taylor series about x_n, then equating and collecting like terms yields:

$$C_0 y(x_n) + C_1 h y'(x_n) + C_2 h^2 y''(x_n) + \cdots = 0. \quad (5)$$

where,

$$\left.\begin{aligned} C_0 &= \alpha_0 + \alpha_1 + \alpha_2 = 0 \\ C_1 &= -\alpha_0 + \tfrac{1}{2}\alpha_2 - \beta_{\frac{1}{2}}(1-\rho) = 0 \\ C_2 &= \tfrac{1}{2}\alpha_0 + \tfrac{1}{8}\alpha_2 - \beta_{\frac{1}{2}}\left(\tfrac{1}{2} + p\right) = 0 \end{aligned}\right\}. \quad (6)$$

In deriving the first point $y_{n+\frac{1}{2}}$ the coefficient α_2 is normalized to 1. Solving the set of equation (6) simultaneously for the values of α_j and β_j gives:

Table 1: the coefficient of the first off-step point formula

α_0	α_1	α_2	$\beta_{\frac{1}{2}}$
$\frac{1}{4}\frac{5\rho+1}{\rho+2}$	$-\frac{9}{4}\frac{\rho+1}{\rho+2}$	1	$\frac{3}{4(\rho+2)}$

Substituting these values in equation (3), we obtain:

$$y_{n+\frac{1}{2}} = -\frac{1}{4}\frac{5\rho+1}{\rho+2} y_{n-1} + \frac{9}{4}\frac{\rho+1}{\rho+2} y_n + \frac{3}{4(\rho+2)} h f_{n+\frac{1}{2}} - \frac{3}{4(\rho+2)} \rho h f_{n-1} \quad (7)$$

Therefore, the same procedures are applied for the derivation of first point, second off-step point, and second point respectively, we obtained the formulae of the remaining points as:

$$\left.\begin{aligned} y_{n+1} &= -\frac{1}{3}\frac{11\rho-2}{\rho+14}y_{n-1} + \frac{2(\rho-4)}{\rho+14}y_n + \frac{8}{3}\frac{\rho+8}{\rho+14}y_{n+\frac{1}{2}} + \frac{4}{\rho+14}hf_{n+1} - \frac{4}{\rho+14}\rho hf_{n-\frac{1}{2}} \\ y_{n+\frac{3}{2}} &= \frac{3}{2}\frac{\rho+1}{4\rho-61}y_{n-1} + \frac{5(8\rho-5)}{4\rho-61}y_n - \frac{15(4\rho-5)}{4\rho-61}y_{n+\frac{1}{2}} + \frac{45}{2}\frac{\rho-5}{4\rho-61}y_{n+1} - \frac{15}{4\rho-61}hf_{n+\frac{3}{2}} + \frac{15}{4\rho-61}\rho hf_n \\ y_{n+2} &= -\frac{1}{5}\frac{\rho+4}{\rho-54}y_{n-1} + \frac{9(\rho+2)}{\rho-54}y_n + \frac{4(3\rho-16)}{\rho-54}y_{n+\frac{1}{2}} - \frac{27(\rho-4)}{\rho-54}y_{n+1} + \frac{36}{5}\frac{\rho-16}{\rho-54}y_{n+\frac{3}{2}} - \frac{12}{\rho-54}hf_{n+2} \\ &\quad + \frac{12}{\rho-54}\rho hf_{n+\frac{1}{2}} \end{aligned}\right\} \quad (8)$$

By combining the formulae in equations (7) and (8), we have obtained a two-point block hybrid (AHBBDF) method for solving stiff systems of ordinary differential equations, given by:

$$\left.\begin{aligned} y_{n+\frac{1}{2}} &= -\frac{1}{4}\frac{5\rho+1}{\rho+2}y_{n-1} + \frac{9}{4}\frac{\rho+1}{\rho+2}y_n + \frac{3}{4(\rho+2)}hf_{n+\frac{1}{2}} - \frac{3}{4(\rho+2)}\rho hf_{n-1} \\ y_{n+1} &= -\frac{1}{3}\frac{11\rho-2}{\rho+14}y_{n-1} + \frac{2(\rho-4)}{\rho+14}y_n + \frac{8}{3}\frac{\rho+8}{\rho+14}y_{n+\frac{1}{2}} + \frac{4}{\rho+14}hf_{n+1} - \frac{4}{\rho+14}\rho hf_{n-\frac{1}{2}} \\ y_{n+\frac{3}{2}} &= \frac{3}{2}\frac{\rho+1}{4\rho-61}y_{n-1} + \frac{5(8\rho-5)}{4\rho-61}y_n - \frac{15(4\rho-5)}{4\rho-61}y_{n+\frac{1}{2}} + \frac{45}{2}\frac{\rho-5}{4\rho-61}y_{n+1} - \frac{15}{4\rho-61}hf_{n+\frac{3}{2}} + \frac{15}{4\rho-61}\rho hf_n \\ y_{n+2} &= -\frac{1}{5}\frac{\rho+4}{\rho-54}y_{n-1} + \frac{9(\rho+2)}{\rho-54}y_n + \frac{4(3\rho-16)}{\rho-54}y_{n+\frac{1}{2}} - \frac{27(\rho-4)}{\rho-54}y_{n+1} + \frac{36}{5}\frac{\rho-16}{\rho-54}y_{n+\frac{3}{2}} - \frac{12}{\rho-54}hf_{n+2} \\ &\quad + \frac{12}{\rho-54}\rho hf_{n+\frac{1}{2}} \end{aligned}\right\} \quad (9)$$

For zero and absolute stability of the method, the parameter ρ is selected from the interval $(-1, 1)$ as suggested in Suleiman et al. (2014). Choosing $\rho = 0$ and $\rho = \frac{1}{5}$ and substituting these values into equation (9), we obtain the diagonally implicit two-point block backward differentiation formula with off-step points (DI2BBDFO) developed by Musa *et al.* (2022) and the diagonally implicit extended two-point super class of block backward differentiation formula with off-step points (DIE2OSBBDF) developed by Alhassan and Musa (2023) as follows:

For $\rho = 0$:

$$\left.\begin{aligned} y_{n+\frac{1}{2}} &= -\frac{1}{8}y_{n-1} + \frac{9}{8}y_n + \frac{3}{8}hf_{n+\frac{1}{2}} \\ y_{n+1} &= \frac{1}{21}y_{n-1} - \frac{4}{7}y_n + \frac{32}{21}y_{n+\frac{1}{2}} + \frac{2}{7}hf_{n+1} \\ y_{n+\frac{3}{2}} &= -\frac{3}{122}y_{n-1} + \frac{25}{61}y_n - \frac{75}{61}y_{n+\frac{1}{2}} + \frac{225}{122}y_{n+1} + \frac{15}{61}hf_{n+\frac{3}{2}} \\ y_{n+2} &= \frac{2}{135}y_{n-1} - \frac{1}{3}y_n + \frac{32}{27}y_{n+\frac{1}{2}} - 2y_{n+1} + \frac{32}{15}y_{n+\frac{3}{2}} + \frac{2}{9}hf_{n+2} \end{aligned}\right\} \quad (10)$$

For $\rho = \frac{1}{5}$:

$$\left.\begin{aligned} y_{n+\frac{1}{2}} &= -\frac{5}{22}y_{n-1} + \frac{27}{22}y_n + \frac{15}{44}hf_{n+\frac{1}{2}} - \frac{3}{44}hf_{n-1} \\ y_{n+1} &= -\frac{1}{213}y_{n-1} - \frac{38}{71}y_n + \frac{328}{213}y_{n+\frac{1}{2}} + \frac{20}{71}hf_{n+1} - \frac{4}{71}hf_{n-\frac{1}{2}} \\ y_{n+\frac{3}{2}} &= -\frac{9}{301}y_{n-1} + \frac{85}{301}y_n - \frac{45}{43}y_{n+\frac{1}{2}} + \frac{540}{301}y_{n+1} + \frac{75}{301}hf_{n+\frac{3}{2}} - \frac{15}{301}hf_n \\ y_{n+2} &= \frac{21}{1345}y_{n-1} - \frac{99}{269}y_n + \frac{308}{269}y_{n+\frac{1}{2}} - \frac{513}{269}y_{n+1} + \frac{2844}{1345}y_{n+\frac{3}{2}} + \frac{60}{269}hf_{n+2} - \frac{12}{269}hf_{n+\frac{1}{2}} \end{aligned}\right\} \quad (11)$$

In the paper, we consider a new proposed two-point diagonally implicit block BDF with off-step points method when the value of the free parameter ρ is taken to be -1/2 and substituted in (9) as given below:

For $\rho = -\frac{1}{2}$:

$$\left.\begin{aligned} y_{n+\frac{1}{2}} &= \frac{1}{4}y_{n-1} + \frac{3}{4}y_n + \frac{1}{2}hf_{n+\frac{1}{2}} + \frac{1}{4}hf_{n-1} \\ y_{n+1} &= \frac{5}{27}y_{n-1} - \frac{2}{3}y_n + \frac{40}{27}y_{n+\frac{1}{2}} + \frac{8}{27}hf_{n+1} + \frac{4}{27}hf_{n-\frac{1}{2}} \\ y_{n+\frac{3}{2}} &= -\frac{1}{84}y_{n-1} + \frac{5}{7}y_n - \frac{5}{3}y_{n+\frac{1}{2}} + \frac{55}{28}y_{n+1} + \frac{5}{21}hf_{n+\frac{3}{2}} + \frac{5}{42}hf_n \\ y_{n+2} &= \frac{7}{545}y_{n-1} - \frac{27}{109}y_n + \frac{140}{109}y_{n+\frac{1}{2}} - \frac{243}{109}y_{n+1} + \frac{1188}{545}y_{n+\frac{3}{2}} + \frac{24}{109}hf_{n+2} + \frac{12}{109}hf_{n+\frac{1}{2}} \end{aligned}\right\} \quad (12)$$

However, it should be noted that throughout this paper, the necessary and sufficient conditions for convergence and stability analysis of the method (3) are generalised in terms of ρ as can be seen in the subsequent sections.

Derivation of Order and Error Constant of the AHBBDF Method

In this section, the detailed derivation of order and its error constant of the two-point block hybrid (AHBBDF) method corresponding to the equations in (9) are presented. To derive the order of the method, the formulae (9) can also be represented as:

$$\left.\begin{aligned} \frac{1}{4}\frac{5\rho+1}{\rho+2}y_{n-1} - \frac{9}{4}\frac{\rho+1}{\rho+2}y_n + y_{n+\frac{1}{2}} &= \frac{3}{4(\rho+2)}hf_{n+\frac{1}{2}} - \frac{3}{4(\rho+2)}\rho hf_{n-1} \\ \frac{1}{3}\frac{11\rho-2}{\rho+14}y_{n-1} - \frac{2(\rho-4)}{\rho+14}y_n - \frac{8}{3}\frac{\rho+8}{\rho+14}y_{n+\frac{1}{2}} + y_{n+1} &= \frac{4}{\rho+14}hf_{n+1} - \frac{4}{\rho+14}\rho hf_{n-\frac{1}{2}} \\ -\frac{3}{2}\frac{\rho+1}{4\rho-61}y_{n-1} - \frac{5(8\rho-5)}{4\rho-61}y_n + \frac{15(4\rho-5)}{4\rho-61}y_{n+\frac{1}{2}} - \frac{45}{2}\frac{\rho-5}{4\rho-61}y_{n+1} + y_{n+\frac{3}{2}} &= -\frac{15}{4\rho-61}hf_{n+\frac{3}{2}} + \frac{15}{4\rho-61}\rho hf_n \\ \frac{1}{5}\frac{\rho+4}{\rho-54}y_{n-1} - \frac{9(\rho+2)}{\rho-54}y_n - \frac{4(3\rho-16)}{\rho-54}y_{n+\frac{1}{2}} + \frac{27(\rho-4)}{\rho-54}y_{n+1} - \frac{36}{5}\frac{\rho-16}{\rho-54}y_{n+\frac{3}{2}} + y_{n+2} &= -\frac{12}{\rho-54}hf_{n+2} + \frac{12}{\rho-54}\rho hf_{n+\frac{1}{2}} \end{aligned}\right\} \quad (13)$$

The formulae (13) can also be expressed in matrix form as:

$$\begin{bmatrix} 0 & \frac{1}{4}\frac{5\rho+1}{\rho+2} & 0 & -\frac{9}{4}\frac{\rho+1}{\rho+2} \\ 0 & \frac{1}{3}\frac{11\rho-2}{\rho+14} & 0 & -\frac{2(\rho-4)}{\rho+14} \\ 0 & -\frac{3}{2}\frac{\rho+1}{4\rho-61} & 0 & -\frac{5(8\rho-5)}{4\rho-61} \\ 0 & \frac{1}{5}\frac{\rho+4}{\rho-54} & 0 & -\frac{9(\rho+2)}{\rho-54} \end{bmatrix} \begin{bmatrix} y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{8}{3}\frac{\rho+8}{\rho+14} & 1 & 0 & 0 \\ \frac{15(4\rho-5)}{4\rho-61} & -\frac{45}{2}\frac{\rho-5}{4\rho-61} & 1 & 0 \\ -\frac{4(3\rho-16)}{\rho-54} & \frac{27(\rho-4)}{\rho-54} & -\frac{36}{5}\frac{\rho-16}{\rho-54} & 1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix}$$

$$= h\begin{bmatrix} 0 & -\frac{3\rho}{4(\rho+2)} & 0 & 0 \\ 0 & 0 & -\frac{4\rho}{\rho+14} & 0 \\ 0 & 0 & 0 & \frac{15\rho}{4\rho-61} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{bmatrix} + h\begin{bmatrix} \frac{3}{4(\rho+2)} & 0 & 0 & 0 \\ 0 & \frac{4}{\rho+14} & 0 & 0 \\ 0 & 0 & -\frac{15}{4\rho-61} & 0 \\ \frac{12\rho}{\rho-54} & 0 & 0 & -\frac{12}{\rho-54} \end{bmatrix} \begin{bmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{bmatrix} \quad (14)$$

Let

$$\gamma_0=\begin{bmatrix}0\\0\\0\\0\end{bmatrix},\gamma_1=\begin{bmatrix}\frac{1}{4}\frac{5\rho+1}{\rho+2}\\ \frac{1}{3}\frac{11\rho-2}{\rho+14}\\ -\frac{3}{2}\frac{\rho+1}{4\rho-61}\\ \frac{1}{5}\frac{\rho+4}{\rho-54}\end{bmatrix},\gamma_2=\begin{bmatrix}0\\0\\0\\0\end{bmatrix},\gamma_3=\begin{bmatrix}-\frac{9}{4}\frac{\rho+1}{\rho+2}\\ -\frac{2(\rho-4)}{\rho+14}\\ -\frac{5(8\rho-5)}{4\rho-61}\\ -\frac{9(\rho+2)}{\rho-54}\end{bmatrix},\gamma_4=\begin{bmatrix}1\\ -\frac{8}{3}\frac{\rho+8}{\rho+14}\\ \frac{15(4\rho-5)}{4\rho-61}\\ -\frac{4(3\rho-16)}{\rho-54}\end{bmatrix},\gamma_5=\begin{bmatrix}0\\1\\ -\frac{45}{2}\frac{\rho-5}{4\rho-61}\\ \frac{27(\rho-4)}{\rho-54}\end{bmatrix},$$

$$\gamma_6=\begin{bmatrix}0\\0\\1\\ -\frac{36}{5}\frac{\rho-16}{\rho-54}\end{bmatrix},\gamma_7=\begin{bmatrix}0\\0\\0\\1\end{bmatrix},\psi_0=\begin{bmatrix}0\\0\\0\\0\end{bmatrix},\psi_1=\begin{bmatrix}-\frac{3\rho}{4(\rho+2)}\\0\\0\\0\end{bmatrix},$$

$$\psi_2=\begin{bmatrix}0\\ -\frac{4\rho}{\rho+14}\\0\\0\end{bmatrix},\psi_3=\begin{bmatrix}0\\0\\ \frac{15\rho}{4\rho-61}\\0\end{bmatrix},\psi_4=\begin{bmatrix}\frac{3}{4(\rho+2)}\\0\\0\\ \frac{12\rho}{\rho-54}\end{bmatrix},\psi_5=\begin{bmatrix}0\\ \frac{4}{\rho+14}\\0\\0\end{bmatrix},\psi_6=\begin{bmatrix}0\\0\\ -\frac{15}{4\rho-61}\\0\end{bmatrix},\psi_7=\begin{bmatrix}0\\0\\0\\ -\frac{12}{\rho-54}\end{bmatrix}$$

Definition 1 (Order): the order of the two-point block hybrid method (9) and its associated linear difference operator given by:

$$L\{y(x),h\}=\sum_{j=0}^{k}\left[\gamma_j y\left(x+j\frac{h}{2}\right)-h\psi_j y'\left(x+j\frac{h}{2}\right)\right] \tag{15}$$

is the unique integer p such that if $A_q=0$, $q=0(1)p$ and $A_{P+1}\neq 0$; where $A_q$are constant (column) matrices defined by:

$$\left.\begin{array}{c} A_0=\gamma_0+\gamma_1+\gamma_2+\cdots+\gamma_k \\ A_1=\gamma_1+2\gamma_2+\cdots+k\gamma_k-2(\psi_0+\psi_1+\psi_2+\cdots+\psi_k) \\ \cdot \\ \cdot \\ \cdot \\ A_q=\frac{1}{q!}(\gamma_1+2^q\gamma_2+\cdots+k^q\gamma_k)-\frac{2}{(q-1)!}(\psi_1+2^{q-1}\psi_2+\cdots+k^{q-1}\psi_k) \end{array}\right\} \tag{16}$$

$$q=2,3,\ldots\ldots\ldots$$

For $q=0(1)6$, we have

$$A_0=\sum_{j=0}^{7}\gamma_j=\gamma_0+\gamma_1+\gamma_2+\gamma_3+\gamma_4+\gamma_5+\gamma_6+\gamma_7$$

$$
=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+\begin{bmatrix}\frac{1}{4}\frac{5\rho+1}{\rho+2}\\ \frac{1}{3}\frac{11\rho-2}{\rho+14}\\ -\frac{3}{2}\frac{\rho+1}{4\rho-61}\\ \frac{1}{5}\frac{\rho+4}{\rho-54}\end{bmatrix}+\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+\begin{bmatrix}-\frac{9}{4}\frac{\rho+1}{\rho+2}\\ -\frac{2(\rho-4)}{\rho+14}\\ -\frac{5(8\rho-5)}{4\rho-61}\\ -\frac{9(\rho+2)}{\rho-54}\end{bmatrix}+\begin{bmatrix}1\\ -\frac{8}{3}\frac{\rho+8}{\rho+14}\\ \frac{15(4\rho-5)}{4\rho-61}\\ -\frac{4(3\rho-16)}{\rho-54}\end{bmatrix}+\begin{bmatrix}0\\1\\ -\frac{45}{2}\frac{\rho-5}{4\rho-61}\\ \frac{27(\rho-4)}{\rho-54}\end{bmatrix}+\begin{bmatrix}0\\0\\1\\ -\frac{36}{5}\frac{\rho-16}{\rho-54}\end{bmatrix}+
$$

$$
\begin{bmatrix}0\\0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}
$$

$$
A_1=\sum_{j=0}^{7}(j\gamma_j)-2\sum_{j=0}^{7}\psi_j=((0)\gamma_0+(1)\gamma_1+(2)\gamma_2+(3)\gamma_3+(4)\gamma_4+(5)\gamma_5+(6)\gamma_6+(7)\gamma_7)
$$

$$
-2(\psi_0+\psi_1+\psi_2+\psi_3+\psi_4+\psi_5+\psi_6+\psi_7)
$$

$$
=\left[(0)\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)\begin{bmatrix}\frac{1}{4}\frac{5\rho+1}{\rho+2}\\ \frac{1}{3}\frac{11\rho-2}{\rho+14}\\ -\frac{3}{2}\frac{\rho+1}{4\rho-61}\\ \frac{1}{5}\frac{\rho+4}{\rho-54}\end{bmatrix}+(2)\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(3)\begin{bmatrix}-\frac{9}{4}\frac{\rho+1}{\rho+2}\\ -\frac{2(\rho-4)}{\rho+14}\\ -\frac{5(8\rho-5)}{4\rho-61}\\ -\frac{9(\rho+2)}{\rho-54}\end{bmatrix}+(4)\begin{bmatrix}1\\ -\frac{8}{3}\frac{\rho+8}{\rho+14}\\ \frac{15(4\rho-5)}{4\rho-61}\\ -\frac{4(3\rho-16)}{\rho-54}\end{bmatrix}+(5)\begin{bmatrix}0\\1\\ -\frac{45}{2}\frac{\rho-5}{4\rho-61}\\ \frac{27(\rho-4)}{\rho-54}\end{bmatrix}+(6)\begin{bmatrix}0\\0\\1\\ -\frac{36}{5}\frac{\rho-16}{\rho-54}\end{bmatrix}+(7)\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right]
$$

$$
-2\left[\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+\begin{bmatrix}-\frac{3\rho}{4(\rho+2)}\\0\\0\\0\end{bmatrix}+\begin{bmatrix}0\\ -\frac{4\rho}{\rho+14}\\0\\0\end{bmatrix}+\begin{bmatrix}0\\0\\ \frac{15\rho}{4\rho-61}\\0\end{bmatrix}+\begin{bmatrix}\frac{3}{4(\rho+2)}\\0\\0\\ \frac{12\rho}{\rho-54}\end{bmatrix}+\begin{bmatrix}0\\ \frac{4}{\rho+14}\\0\\0\end{bmatrix}+\begin{bmatrix}0\\0\\ -\frac{15}{4\rho-61}\\0\end{bmatrix}+\begin{bmatrix}0\\0\\0\\ -\frac{12}{\rho-54}\end{bmatrix}\right]
$$

$$
=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}
$$

$$
A_2=\sum_{j=0}^{7}\frac{(j^2\gamma_j)}{2!}-2\sum_{j=0}^{7}(j\psi_j)=\frac{1}{2!}((0)^2\gamma_0+(1)^2\gamma_1+(2)^2\gamma_2+(3)^2\gamma_3+(4)^2\gamma_4+(5)^2\gamma_5+(6)^2\gamma_6+(7)^2\gamma_7)
$$

$$
-2((0)\psi_0+(1)\psi_1+(2)\psi_2+(3)\psi_3+(4)\psi_4+(5)\psi_5+(6)\psi_6+(7)\psi_7)
$$

$$
=\frac{1}{2!}\left[(0)^2\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)^2\begin{bmatrix}\frac{1}{4}\frac{5\rho+1}{\rho+2}\\ \frac{1}{3}\frac{11\rho-2}{\rho+14}\\ -\frac{3}{2}\frac{\rho+1}{4\rho-61}\\ \frac{1}{5}\frac{\rho+4}{\rho-54}\end{bmatrix}+(2)^2\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(3)^2\begin{bmatrix}-\frac{9}{4}\frac{\rho+1}{\rho+2}\\ -\frac{2(\rho-4)}{\rho+14}\\ -\frac{5(8\rho-5)}{4\rho-61}\\ -\frac{9(\rho+2)}{\rho-54}\end{bmatrix}+(4)^2\begin{bmatrix}1\\ -\frac{8}{3}\frac{\rho+8}{\rho+14}\\ \frac{15(4\rho-5)}{4\rho-61}\\ -\frac{4(3\rho-16)}{\rho-54}\end{bmatrix}+(5)^2\begin{bmatrix}0\\1\\ -\frac{45}{2}\frac{\rho-5}{4\rho-61}\\ \frac{27(\rho-4)}{\rho-54}\end{bmatrix}+(6)^2\begin{bmatrix}0\\0\\1\\ -\frac{36}{5}\frac{\rho-16}{\rho-54}\end{bmatrix}+(7)^2\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right]
$$

$$
-2\left[(0)\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)\begin{bmatrix}-\frac{3}{4(\rho+2)}\\0\\0\\0\end{bmatrix}+(2)\begin{bmatrix}0\\ -\frac{4\rho}{\rho+14}\\0\\0\end{bmatrix}+(3)\begin{bmatrix}0\\0\\ \frac{15\rho}{4\rho-61}\\0\end{bmatrix}+(4)\begin{bmatrix}\frac{3}{4(\rho+2)}\\0\\0\\ \frac{12\rho}{\rho-54}\end{bmatrix}+(5)\begin{bmatrix}0\\ \frac{4}{\rho+14}\\0\\0\end{bmatrix}+(6)\begin{bmatrix}0\\0\\ -\frac{15}{4\rho-61}\\0\end{bmatrix}+(7)\begin{bmatrix}0\\0\\0\\ -\frac{12}{\rho-54}\end{bmatrix}\right]
$$

$$
=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}
$$

$$
A_3=\sum_{j=0}^{7}\frac{\left(j^3\gamma_j\right)}{3!}-2\sum_{j=0}^{7}\frac{\left(j^2\psi_j\right)}{2!}=\frac{1}{3!}\left((0)^3\gamma_0+(1)^3\gamma_1+(2)^3\gamma_2+(3)^3\gamma_3+(4)^3\gamma_4+(5)^3\gamma_5+(6)^3\gamma_6+(7)^3\gamma_7\right)
$$

$$
-\frac{2}{2!}\left((0)^2\psi_0+(1)^2\psi_1+(2)^2\psi_2+(3)^2\psi_3+(4)^2\psi_4+(5)^2\psi_5+(6)^2\psi_6+(7)^2\psi_7\right)
$$

$$
=\frac{1}{3!}\left[(0)^3\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)^3\begin{bmatrix}\frac{1}{4}\frac{5\rho+1}{\rho+2}\\ \frac{1}{3}\frac{11\rho-2}{\rho+14}\\ -\frac{3}{2}\frac{\rho+1}{4\rho-61}\\ \frac{1}{5}\frac{\rho+4}{\rho-54}\end{bmatrix}+(2)^3\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(3)^3\begin{bmatrix}-\frac{9}{4}\frac{\rho+1}{\rho+2}\\ -\frac{2(\rho-4)}{\rho+14}\\ -\frac{5(8\rho-5)}{4\rho-61}\\ -\frac{9(\rho+2)}{\rho-54}\end{bmatrix}+(4)^3\begin{bmatrix}1\\ -\frac{8}{3}\frac{\rho+8}{\rho+14}\\ \frac{15(4\rho-5)}{4\rho-61}\\ -\frac{4(3\rho-16)}{\rho-54}\end{bmatrix}+(5)^3\begin{bmatrix}0\\1\\ -\frac{45}{2}\frac{\rho-5}{4\rho-61}\\ \frac{27(\rho-4)}{\rho-54}\end{bmatrix}+(6)^3\begin{bmatrix}0\\0\\1\\ -\frac{36}{5}\frac{\rho-16}{\rho-54}\end{bmatrix}+(7)^3\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right]
$$

$$
-\frac{2}{2!}\left[(0)^2\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)^2\begin{bmatrix}-\frac{3}{4(\rho+2)}\\0\\0\\0\end{bmatrix}+(2)^2\begin{bmatrix}0\\ -\frac{4\rho}{\rho+14}\\0\\0\end{bmatrix}+(3)^2\begin{bmatrix}0\\0\\ \frac{15\rho}{4\rho-61}\\0\end{bmatrix}+(4)^2\begin{bmatrix}\frac{3}{4(\rho+2)}\\0\\0\\ \frac{12\rho}{\rho-54}\end{bmatrix}+(5)^2\begin{bmatrix}0\\ \frac{4}{\rho+14}\\0\\0\end{bmatrix}+(6)^2\begin{bmatrix}0\\0\\ -\frac{15}{4\rho-61}\\0\end{bmatrix}+(7)^2\begin{bmatrix}0\\0\\0\\ -\frac{12}{\rho-54}\end{bmatrix}\right]
$$

$$
=\begin{bmatrix}\frac{3(-1+2\rho)}{4(\rho+2)}\\0\\0\\0\end{bmatrix}
$$

$$
A_4=\sum_{j=0}^{7}\frac{\left(j^4\gamma_j\right)}{4!}-2\sum_{j=0}^{7}\frac{\left(j^3\psi_j\right)}{3!}=\frac{1}{4!}\left((0)^4\gamma_0+(1)^4\gamma_1+(2)^4\gamma_2+(3)^4\gamma_3+(4)^4\gamma_4+(5)^4\gamma_5+(6)^4\gamma_6+(7)^4\gamma_7\right)
$$

$$
-\frac{2}{3!}\left((0)^3\psi_0+(1)^3\psi_1+(2)^3\psi_2+(3)^3\psi_3+(4)^3\psi_4+(5)^3\psi_5+(6)^3\gamma_6+(7)^3\gamma_7\right)
$$

$$=\frac{1}{4!}\left[(0)^4\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)^4\begin{bmatrix}\frac{1}{4}\frac{5\rho+1}{\rho+2}\\ \frac{1}{3}\frac{11\rho-2}{\rho+14}\\ -\frac{3}{2}\frac{\rho+1}{4\rho-61}\\ \frac{1}{5}\frac{\rho+4}{\rho-54}\end{bmatrix}+(2)^4\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(3)^4\begin{bmatrix}-\frac{9}{4}\frac{\rho+1}{\rho+2}\\ -\frac{2(\rho-4)}{\rho+14}\\ -\frac{5(8\rho-5)}{4\rho-61}\\ -\frac{9(\rho+2)}{\rho-54}\end{bmatrix}+(4)^4\begin{bmatrix}1\\ -\frac{8}{3}\frac{\rho+8}{\rho+14}\\ \frac{15(4\rho-5)}{4\rho-61}\\ -\frac{4(3\rho-16)}{\rho-54}\end{bmatrix}+(5)^4\begin{bmatrix}0\\1\\ -\frac{45}{2}\frac{\rho-5}{4\rho-61}\\ \frac{27(\rho-4)}{\rho-54}\end{bmatrix}+(6)^4\begin{bmatrix}0\\0\\1\\ -\frac{36}{5}\frac{\rho-16}{\rho-54}\end{bmatrix}+(7)^4\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right]$$

$$-\frac{2}{3!}\left[(0)^3\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)^3\begin{bmatrix}-\frac{3}{4(\rho+2)}\\0\\0\\0\end{bmatrix}+(2)^3\begin{bmatrix}0\\ -\frac{4\rho}{\rho+14}\\0\\0\end{bmatrix}+(3)^3\begin{bmatrix}0\\0\\ \frac{15\rho}{4\rho-61}\\0\end{bmatrix}+(4)^3\begin{bmatrix}\frac{3}{4(\rho+2)}\\0\\0\\ \frac{12\rho}{\rho-54}\end{bmatrix}+(5)^3\begin{bmatrix}0\\ \frac{4}{\rho+14}\\0\\0\end{bmatrix}+(6)^3\begin{bmatrix}0\\0\\ -\frac{15}{4\rho-61}\\0\end{bmatrix}+(7)^3\begin{bmatrix}0\\0\\0\\ -\frac{12}{\rho-54}\end{bmatrix}\right]$$

$$=\begin{bmatrix}\frac{9(-2+3\rho)}{8(\rho+2)}\\ \frac{-8+5\rho}{3(\rho+14)}\\ 0\\ 0\end{bmatrix}$$

$$A_5=\sum_{j=0}^{7}\frac{(j^5\gamma_j)}{5!}-2\sum_{j=0}^{7}\frac{(j^4\psi_j)}{4!}=\frac{1}{5!}\left((0)^5\gamma_0+(1)^5\gamma_1+(2)^5\gamma_2+(3)^5\gamma_3+(4)^5\gamma_4+(5)^5\gamma_5+(6)^5\gamma_6+(7)^5\gamma_7\right)$$
$$-\frac{2}{4!}\left((0)^4\psi_0+(1)^4\psi_1+(2)^4\psi_2+(3)^4\psi_3+(4)^4\psi_4+(5)^4\psi_5+(6)^4\psi_6+(7)^4\psi_7\right)$$

$$=\frac{1}{5!}\left[(0)^5\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)^5\begin{bmatrix}\frac{1}{4}\frac{5\rho+1}{\rho+2}\\ \frac{1}{3}\frac{11\rho-2}{\rho+14}\\ -\frac{3}{2}\frac{\rho+1}{4\rho-61}\\ \frac{1}{5}\frac{\rho+4}{\rho-54}\end{bmatrix}+(2)^5\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(3)^5\begin{bmatrix}-\frac{9}{4}\frac{\rho+1}{\rho+2}\\ -\frac{2(\rho-4)}{\rho+14}\\ -\frac{5(8\rho-5)}{4\rho-61}\\ -\frac{9(\rho+2)}{\rho-54}\end{bmatrix}+(4)^5\begin{bmatrix}1\\ -\frac{8}{3}\frac{\rho+8}{\rho+14}\\ \frac{15(4\rho-5)}{4\rho-61}\\ -\frac{4(3\rho-16)}{\rho-54}\end{bmatrix}+(5)^5\begin{bmatrix}0\\1\\ -\frac{45}{2}\frac{\rho-5}{4\rho-61}\\ \frac{27(\rho-4)}{\rho-54}\end{bmatrix}+(6)^5\begin{bmatrix}0\\0\\1\\ -\frac{36}{5}\frac{\rho-16}{\rho-54}\end{bmatrix}+(7)^5\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right]$$

$$-\frac{2}{4!}\left[(0)^4\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)^4\begin{bmatrix}-\frac{3}{4(\rho+2)}\\0\\0\\0\end{bmatrix}+(2)^4\begin{bmatrix}0\\ -\frac{4\rho}{\rho+14}\\0\\0\end{bmatrix}+(3)^4\begin{bmatrix}0\\0\\ \frac{15\rho}{4\rho-61}\\0\end{bmatrix}+(4)^4\begin{bmatrix}\frac{3}{4(\rho+2)}\\0\\0\\ \frac{12\rho}{\rho-54}\end{bmatrix}+(5)^4\begin{bmatrix}0\\ \frac{4}{\rho+14}\\0\\0\end{bmatrix}+(6)^4\begin{bmatrix}0\\0\\ -\frac{15}{4\rho-61}\\0\end{bmatrix}+(7)^4\begin{bmatrix}0\\0\\0\\ -\frac{12}{\rho-54}\end{bmatrix}\right]$$

$$=\begin{bmatrix}\frac{9(-31+36\rho)}{80(\rho+2)}\\ \frac{-48+23\rho}{5(\rho+14)}\\ \frac{3(5+2\rho)}{2(4\rho-61)}\\ 0\end{bmatrix}$$

$$A_6=\sum_{j=0}^{7}\frac{(j^6\gamma_j)}{6!}-2\sum_{j=0}^{7}\frac{(j^5\psi_j)}{5!}=\frac{1}{6!}\left((0)^6\gamma_0+(1)^6\gamma_1+(2)^6\gamma_2+(3)^6\gamma_3+(4)^6\gamma_4+(5)^6\gamma_5+(6)^6\gamma_6+(7)^6\gamma_7\right)$$

$$-\frac{2}{5!}\left((0)^5\psi_0+(1)^5\psi_1+(2)^5\psi_2+(3)^5\psi_3+(4)^5\psi_4+(5)^5\psi_5+(6)^5\psi_6+(7)^5\psi_7\right)$$

$$=\frac{1}{6!}\left[(0)^6\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)^6\begin{bmatrix}\frac{1}{4}\frac{5\rho+1}{\rho+2}\\\frac{1}{3}\frac{11\rho-2}{\rho+14}\\-\frac{3}{2}\frac{\rho+1}{4\rho-61}\\\frac{1}{5}\frac{\rho+4}{\rho-54}\end{bmatrix}+(2)^6\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(3)^6\begin{bmatrix}-\frac{9}{4}\frac{\rho+1}{\rho+2}\\-\frac{2(\rho-4)}{\rho+14}\\-\frac{5(8\rho-5)}{4\rho-61}\\-\frac{9(\rho+2)}{\rho-54}\end{bmatrix}+(4)^6\begin{bmatrix}1\\-\frac{8}{3}\frac{\rho+8}{\rho+14}\\\frac{15(4\rho-5)}{4\rho-61}\\-\frac{4(3\rho-16)}{\rho-54}\end{bmatrix}+(5)^6\begin{bmatrix}0\\1\\-\frac{45}{2}\frac{\rho-5}{4\rho-61}\\\frac{27(\rho-4)}{\rho-54}\end{bmatrix}+(6)^6\begin{bmatrix}0\\0\\1\\-\frac{36}{5}\frac{\rho-16}{\rho-54}\end{bmatrix}+(7)^6\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right]$$

$$-\frac{2}{5!}\left[(0)^5\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)^5\begin{bmatrix}-\frac{3}{4(\rho+2)}\\0\\0\\0\end{bmatrix}+(2)^5\begin{bmatrix}0\\-\frac{4\rho}{\rho+14}\\0\\0\end{bmatrix}+(3)^5\begin{bmatrix}0\\0\\\frac{15\rho}{4\rho-61}\\0\end{bmatrix}+(4)^5\begin{bmatrix}\frac{3}{4(\rho+2)}\\0\\0\\\frac{12\rho}{\rho-54}\end{bmatrix}+(5)^5\begin{bmatrix}0\\\frac{4}{\rho+14}\\0\\0\end{bmatrix}+(6)^5\begin{bmatrix}0\\0\\-\frac{15}{4\rho-61}\\0\end{bmatrix}+(7)^5\begin{bmatrix}0\\0\\0\\-\frac{12}{\rho-54}\end{bmatrix}\right]$$

$$=\begin{bmatrix}\frac{(-148+137\rho)}{40(\rho+2)}\\\frac{-800+299\rho}{45(\rho+14)}\\\frac{(125+44\rho)}{4(4\rho-61)}\\\frac{3(\rho+8)}{5(\rho-54)}\end{bmatrix}$$

From the previous definition 1, since $A_0=A_1=A_2=A_3=A_4=A_5=0$ and $A_6\neq 0$, then we deduced that the two-point block hybrid method is of order 5 with an error constant $A_6\neq 0$given by:

$$A_6=\begin{bmatrix}\frac{(-148+137\rho)}{40(\rho+2)}\\\frac{-800+299\rho}{45(\rho+14)}\\\frac{(125+44\rho)}{4(4\rho-61)}\\\frac{3(\rho+8)}{5(\rho-54)}\end{bmatrix}\neq\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

The specific error constants corresponding to $\rho=0$, $\rho=\frac{1}{5}$ and $\rho=-\frac{1}{2}$ are given by

For $\rho=0$

$$A_6=\begin{bmatrix}-\frac{37}{20}\\-\frac{80}{63}\\-\frac{125}{244}\\-\frac{4}{45}\end{bmatrix}\neq\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

For $\rho=\frac{1}{5}$

$$A_6 = \begin{bmatrix} -\frac{603}{440} \\ -\frac{3701}{3195} \\ -\frac{669}{1204} \\ -\frac{447}{1345} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\rho = -\frac{1}{2}$

$$A_6 = \begin{bmatrix} -\frac{433}{120} \\ -\frac{211}{135} \\ -\frac{103}{252} \\ -\frac{9}{109} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Definition 2 (Consistency): The block implicit method (9) is consistent if it has order at least one. Hence, according to this definition, we conclude that the method is consistent since its order is five which is greater than one.

Stability Analysis of the Method

This section presents the conditions for the stability properties of method in (9) by first introducing the following definitions of zero-stability and A-stability as:

Definition 3 (Zero Stability): The block implicit method (9) is said to be zero stable if no root of the first characteristic polynomial has modulus greater than one and any root with modulus one is simple (Alhassan and Musa *et al.* 2023).

Definition 4 (A-Stability):The block implicit method (9) is said to be *A*-stable if the stability region covers the entire negative left-half plane. (Abasi *et al.* 2014).

The formulae (9) can be cast in matrix form as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{4}\frac{5\rho+1}{\rho+2} & 0 & \frac{9}{4}\frac{\rho+1}{\rho+2} \\ 0 & -\frac{1}{3}\frac{11\rho-2}{\rho+14} & 0 & \frac{2(\rho-4)}{\rho+14} \\ 0 & \frac{3}{2}\frac{\rho+1}{4\rho-61} & 0 & \frac{5(8\rho-5)}{4\rho-61} \\ 0 & -\frac{1}{5}\frac{\rho+4}{\rho-54} & 0 & \frac{9(\rho+2)}{\rho-54} \end{bmatrix} \begin{bmatrix} y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{8}{3}\frac{\rho+8}{\rho+14} & 0 & 0 & 0 \\ -\frac{15(4\rho-5)}{4\rho-61} & \frac{45}{2}\frac{\rho-5}{4\rho-61} & 0 & 0 \\ \frac{4(3\rho-16)}{\rho-54} & -\frac{27(\rho-4)}{\rho-54} & \frac{36}{5}\frac{\rho-16}{\rho-54} & 0 \end{bmatrix}\begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix} + h\begin{bmatrix} 0 & -\frac{3\rho}{4(\rho+2)} & 0 & 0 \\ 0 & 0 & -\frac{4\rho}{\rho+14} & 0 \\ 0 & 0 & 0 & \frac{15\rho}{4\rho-61} \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{bmatrix} +$$

$$h\begin{bmatrix} \frac{3}{4(\rho+2)} & 0 & 0 & 0 \\ 0 & \frac{4}{\rho+14} & 0 & 0 \\ 0 & 0 & -\frac{15}{4\rho-61} & 0 \\ \frac{12\rho}{\rho-54} & 0 & 0 & -\frac{12}{\rho-54} \end{bmatrix}\begin{bmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{bmatrix} \qquad (17)$$

Which is equivalent to

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{8}{3}\frac{\rho+8}{\rho+14} & 1 & 0 & 0 \\ \frac{15(4\rho-5)}{4\rho-61} & -\frac{45}{2}\frac{\rho-5}{4\rho-61} & 1 & 0 \\ -\frac{4(3\rho-16)}{\rho-54} & \frac{27(\rho-4)}{\rho-54} & -\frac{36}{5}\frac{\rho-16}{\rho-54} & 1 \end{bmatrix}\begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{4}\frac{5\rho+1}{\rho+2} & 0 & \frac{9}{4}\frac{\rho+1}{\rho+2} \\ 0 & -\frac{1}{3}\frac{11\rho-2}{\rho+14} & 0 & \frac{2(\rho-4)}{\rho+14} \\ 0 & \frac{3}{2}\frac{\rho+1}{4\rho-61} & 0 & \frac{5(8\rho-5)}{4\rho-61} \\ 0 & -\frac{1}{5}\frac{\rho+4}{\rho-54} & 0 & \frac{9(\rho+2)}{\rho-54} \end{bmatrix}\begin{bmatrix} y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{bmatrix} +$$

$$h\begin{bmatrix} 0 & -\frac{3\rho}{4(\rho+2)} & 0 & 0 \\ 0 & 0 & -\frac{4\rho}{\rho+14} & 0 \\ 0 & 0 & 0 & \frac{15\rho}{4\rho-61} \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{bmatrix} + h\begin{bmatrix} \frac{3}{4(\rho+2)} & 0 & 0 & 0 \\ 0 & \frac{4}{\rho+14} & 0 & 0 \\ 0 & 0 & -\frac{15}{4\rho-61} & 0 \\ \frac{12\rho}{\rho-54} & 0 & 0 & -\frac{12}{\rho-54} \end{bmatrix}\begin{bmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{bmatrix} \qquad (18)$$

Equation (18) can be represented in the following form:

$$A_0 Y_m = A_1 Y_{m-1} + h(B_0 F_{m-1} + B_1 F_m) \qquad (19)$$

where,

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{8}{3}\frac{\rho+8}{\rho+14} & 1 & 0 & 0 \\ \frac{15(4\rho-5)}{4\rho-61} & -\frac{45}{2}\frac{\rho-5}{4\rho-61} & 1 & 0 \\ -\frac{4(3\rho-16)}{\rho-54} & \frac{27(\rho-4)}{\rho-54} & -\frac{36}{5}\frac{\rho-16}{\rho-54} & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & -\frac{1}{4}\frac{5\rho+1}{\rho+2} & 0 & \frac{9}{4}\frac{\rho+1}{\rho+2} \\ 0 & -\frac{1}{3}\frac{11\rho-2}{\rho+14} & 0 & \frac{2(\rho-4)}{\rho+14} \\ 0 & \frac{3}{2}\frac{\rho+1}{4\rho-61} & 0 & \frac{5(8\rho-5)}{4\rho-61} \\ 0 & -\frac{1}{5}\frac{\rho+4}{\rho-54} & 0 & \frac{9(\rho+2)}{\rho-54} \end{bmatrix}$$

$$B_0=\begin{bmatrix}0 & -\dfrac{3\rho}{4(\rho+2)} & 0 & 0\\ 0 & 0 & -\dfrac{4\rho}{\rho+14} & 0\\ 0 & 0 & 0 & \dfrac{15\rho}{4\rho-61}\\ 0 & 0 & 0 & 0\end{bmatrix},\ B_1=\begin{bmatrix}\dfrac{3}{4(\rho+2)} & 0 & 0 & 0\\ 0 & \dfrac{4}{\rho+14} & 0 & 0\\ 0 & 0 & -\dfrac{15}{4\rho-61} & 0\\ \dfrac{12\rho}{\rho-54} & 0 & 0 & -\dfrac{12}{\rho-54}\end{bmatrix}$$

$$Y_m=\begin{bmatrix}y_{n+\frac{1}{2}}\\ y_{n+1}\\ y_{n+\frac{3}{2}}\\ y_{n+2}\end{bmatrix},\ Y_{m-1}=\begin{bmatrix}y_{n-\frac{3}{2}}\\ y_{n-1}\\ y_{n-\frac{1}{2}}\\ y_{n}\end{bmatrix},\ F_{m-1}=\begin{bmatrix}f_{n-\frac{3}{2}}\\ f_{n-1}\\ f_{n-\frac{1}{2}}\\ f_{n}\end{bmatrix},\ F_m=\begin{bmatrix}f_{n+\frac{1}{2}}\\ f_{n+1}\\ f_{n+\frac{3}{2}}\\ f_{n+2}\end{bmatrix}$$

The stability properties of the method is determined by applying the linear test differential equation of the form $y'=\lambda y, \lambda<0$ in (19) to obtain the following equation of the form:

$$A_0Y_m=A_1Y_{m-1}+\lambda h(B_0Y_{m-1}+B_1Y_m) \tag{20}$$

Substituting $\bar{h}=\lambda h$ in the above equation and it leads to

$$(A_0-\bar{h}B_1)Y_m=(A_1+\bar{h}B_0)Y_{m-1} \tag{21}$$

The evaluation of the determinant of the form $det\left((A_0-\bar{h}B_1)-(A_1+\bar{h}B_0)\right)=0$ gives the stability polynomial as

$$-\frac{1}{4}\frac{t}{(\rho+2)(\rho+14)(4\rho-61)(\rho-54)}\left(\begin{array}{c}2160\bar{h}^4\rho^4-2160\bar{h}^4t^3+11520\bar{h}^3\rho^4\\ -14580\bar{h}^3\rho^3t-3384\bar{h}^3\rho^2t^2+2664\bar{h}^3\rho t^3+16056\bar{h}^2\rho^4t\\ -6714\bar{h}^2\rho^3t^2+429\bar{h}^2\rho^2t^3+472\bar{h}\rho^4t^2-176\bar{h}\rho^3t^3-16\rho^4t^3\\ -14436\bar{h}^3\rho^3+34776\bar{h}^3\rho^2t-51336\bar{h}^3\rho t^2+31824\bar{h}^3t^3\\ +26136\bar{h}^2\rho^4-40536\bar{h}^2\rho^3t+72024\bar{h}^2\rho^2t^2-32832\bar{h}^2\rho t^3\\ -20280\bar{h}\rho^4t+17196\bar{h}\rho^3t^2-2831\bar{h}\rho^2t^3-544\rho^4t^2+852\rho^3t^3\\ -173796\bar{h}^2t^3+20960\bar{h}\rho^4+63228\bar{h}\rho^3t-360426\bar{h}\rho^2t^2\\ +133720\bar{h}\rho t^3+560\rho^4t-18312\rho^3t^2+4104\rho^2t^3-1818\bar{h}^2\rho^2\\ -17604\bar{h}^2\rho t+75048\bar{h}^2t^2-47032\bar{h}\rho^3+379680\rho^2t^2\\ -179792\rho t^3-84\bar{h}^2\rho+252\bar{h}^2t-7840\bar{h}\rho^2-112140\bar{h}\rho t\\ +326280\bar{h}t^2-383784\rho^2t+360976\rho t^2-368928t^3-72\bar{h}\rho\\ +1452\bar{h}t-181184\rho t+365760t^2+3168t\end{array}\right)=0 \tag{22}$$

To show that the method (9) is zero-stable, the first characteristics polynomial of the method is therefore determined by substituting the value of $\bar{h}=0$ in (22)

$$\begin{aligned}\pi(t,0)=&-\frac{1}{4}\frac{t}{(\rho+2)(\rho+14)(4\rho-61)(\rho-54)}(-16\rho^4t^3-544\rho^4t^2+852\rho^3t^3+560\rho^4t\\ &-18312\rho^3t^2+4104\rho^2t^3+379680\rho^2t^2-179792\rho t^3-383784\rho^2t\\ &+360976\rho t^2-368928t^3-181184\rho t+365760t^2+3168t)\\ =&\,0\end{aligned} \tag{23}$$

Solving equation (23) for t leads to the following roots:

$$t=0, t=0, t=1, t=-\frac{140\rho^4+4365\rho^3-95946\rho^2-45296\rho+792}{4\rho^4-213\rho^3-1026\rho^2+44948\rho+92232} \tag{24}$$

In order to ensure stability of the method, any suitable value of ρ that may lead to zero stability has to be specifically chosen within the interval $-1<\rho<1$ as in Alhassan and Musa (2023). In this paper, we consider three different values of $\rho = 0$, represents roots in Musa *et al*. (2022), $\rho = \frac{1}{5}$ corresponds to the roots in Alhassan and Musa (2023) while $\rho = -\frac{1}{2}$ represents the new roots that are not in the literature. Thus, by definition 2, the values of t obtained indicated that the two point block hybrid (TPBH) method is zero stable since all modulus of the root is less than one and the root $t = 1$ is simple.

Plotting the stability polynomial (22) in Maple18 environment, we obtain the stability region of the method (10) as:

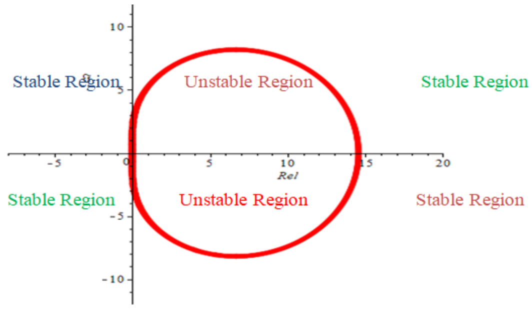

Figure 1: Stability region of the method when $\rho = 0$

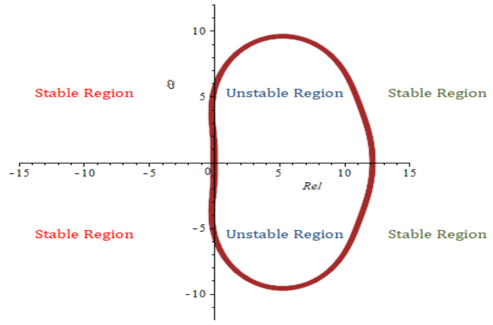

Figure 2: Stability region of the method when $\rho = \frac{1}{5}$

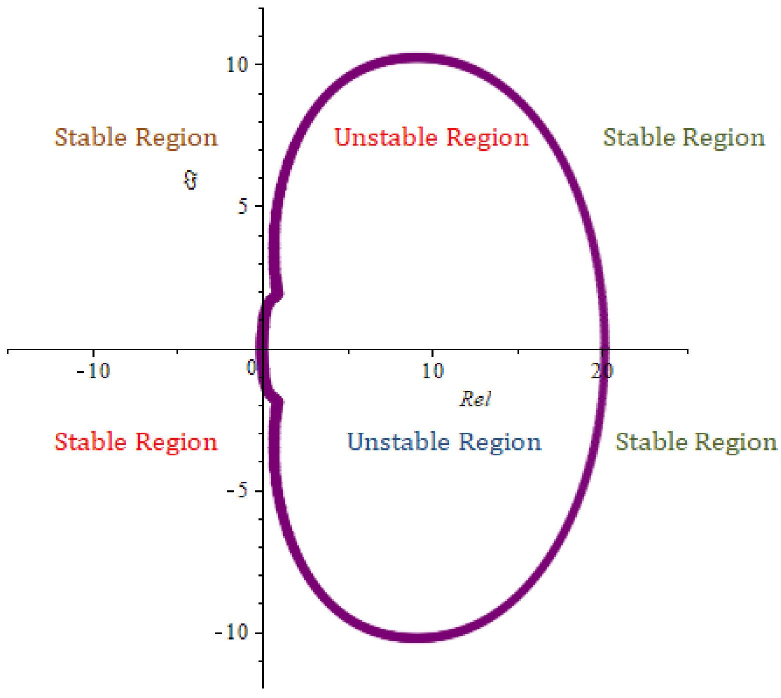

Figure 3: Stability region of the method when $\rho = -\frac{1}{2}$

Thus, by the definition of A-stability, the method (9) is A-stable since the stability regions cover the entire left hand plane, for all the three values of ρ plotted. We therefore conclude that the method is suitable for solving first order stiff initial value problems (IVPs) since the A-stability property is attained by the method.

Theorem 1: the necessary and sufficient conditions for a linear multistep method to be convergent are that to be consistent and zero-stable (Lambert, 1973).

Theorem 2: the two point hybrid diagonally implicit block method (9) is convergent.

Proof:
Having previously satisfied the consistency and zero-stability conditions of the method (9), we therefore conclude that the method is convergent in accordance with theorem 1.

IMPLEMENTATION OF THE METHOD

Newton's iteration is applied for the implementation of the method. We consider the implementation when $\rho = -\frac{1}{2}, \rho = 0$and $\rho = \frac{1}{5}$. The iteration is given below.

Definition 4 (Absolute Error)**:** Let y_i and $y(x_i)$ be the theoretical and approximate solutions of (1). Then the absolute error is defined by

$$(error_i)_t = |(y_i)_t - (y(x_i))_t| \tag{25}$$

The maximum error is defined by:

$$MAXE = \underbrace{max}_{1 \le i \le T} (\underbrace{max(error_i)_t}_{1 \le i \le T}), \tag{26}$$

where T is the number of total steps and N is the number of equations.
Define

$$\left.\begin{aligned} F_{\frac{1}{2}} &= y_{n+\frac{1}{2}} - \frac{3}{4(\rho+2)} hf_{n+\frac{1}{2}} + \frac{3}{4(\rho+2)} \rho hf_{n-1} - \varepsilon_{\frac{1}{2}} \\ F_1 &= y_{n+1} - \frac{8}{3}\frac{\rho+8}{\rho+14} y_{n+\frac{1}{2}} - \frac{4}{\rho+14} hf_{n+1} + \frac{4}{\rho+14} \rho hf_{n-\frac{1}{2}} - \varepsilon_1 \\ F_{\frac{3}{2}} &= y_{n+\frac{3}{2}} + \frac{15(4\rho-5)}{4\rho-61} y_{n+\frac{1}{2}} - \frac{45}{2}\frac{\rho-5}{4\rho-61} y_{n+1} + \frac{15}{4\rho-61} hf_{n+\frac{3}{2}} - \frac{15}{4\rho-61} \rho hf_n - \varepsilon_{\frac{3}{2}} \\ F_2 &= y_{n+2} - \frac{4(3\rho-16)}{\rho-54} y_{n+\frac{1}{2}} + \frac{27(\rho-4)}{\rho-54} y_{n+1} - \frac{36}{5}\frac{\rho-16}{\rho-54} y_{n+\frac{3}{2}} + \frac{12}{\rho-54} hf_{n+2} - \frac{12}{\rho-54} \rho hf_{n+\frac{1}{2}} - \varepsilon_2 \end{aligned}\right\} \tag{27}$$

where,

$$\left.\begin{aligned} \varepsilon_{\frac{1}{2}} &= -\frac{1}{4}\frac{5\rho+1}{\rho+2} y_{n-1} + \frac{9}{4}\frac{\rho+1}{\rho+2} y_n \\ \varepsilon_1 &= -\frac{1}{3}\frac{11\rho-2}{\rho+14} y_{n-1} + \frac{2(\rho-4)}{\rho+14} y_n \\ \varepsilon_{\frac{3}{2}} &= \frac{3}{2}\frac{\rho+1}{4\rho-61} y_{n-1} + \frac{5(8\rho-5)}{4\rho-61} y_n \\ \varepsilon_2 &= -\frac{1}{5}\frac{\rho+4}{\rho-54} y_{n-1} + \frac{9(\rho+2)}{\rho-54} y_n \end{aligned}\right\} \tag{28}$$

are the back values.

Let $y_{n+j}^{(i+1)}$, $j = \frac{1}{2}, 1, \frac{3}{2}, 2$, denote the $(i+1)^{th}$ iterative values of y_{n+j} and define

$$e_{n+j}^{(i+1)} = y_{n+j}^{(i+1)} - y_{n+j}^{(i)}, \qquad j = \frac{1}{2}, 1, \frac{3}{2}, 2 \tag{29}$$

Newton's iteration for the method takes the form:

$$\begin{bmatrix} y_{n+\frac{1}{2}}^{(i+1)} \\ y_{n+1}^{(i+1)} \\ y_{n+\frac{3}{2}}^{(i+1)} \\ y_{n+2}^{(i+1)} \end{bmatrix} = \begin{bmatrix} y_{n+\frac{1}{2}}^{(i)} \\ y_{n+1}^{(i)} \\ y_{n+\frac{3}{2}}^{(i)} \\ y_{n+2}^{(i)} \end{bmatrix} - \begin{bmatrix} \frac{\partial F_{\frac{1}{2}}^{(i)}}{\partial y_{n+\frac{1}{2}}^{(i)}} & \frac{\partial F_{\frac{1}{2}}^{(i)}}{\partial y_{n+1}^{(i)}} & \frac{\partial F_{\frac{1}{2}}^{(i)}}{\partial y_{n+\frac{3}{2}}^{(i)}} & \frac{\partial F_{\frac{1}{2}}^{(i)}}{\partial y_{n+2}^{(i)}} \\ \frac{\partial F_{1}^{(i)}}{\partial y_{n+\frac{1}{2}}^{(i)}} & \frac{\partial F_{1}^{(i)}}{\partial y_{n+1}^{(i)}} & \frac{\partial F_{1}^{(i)}}{\partial y_{n+\frac{3}{2}}^{(i)}} & \frac{\partial F_{1}^{(i)}}{\partial y_{n+2}^{(i)}} \\ \frac{\partial F_{\frac{3}{2}}^{(i)}}{\partial y_{n+\frac{1}{2}}^{(i)}} & \frac{\partial F_{\frac{3}{2}}^{(i)}}{\partial y_{n+1}^{(i)}} & \frac{\partial F_{\frac{3}{2}}^{(i)}}{\partial y_{n+\frac{3}{2}}^{(i)}} & \frac{\partial F_{\frac{3}{2}}^{(i)}}{\partial y_{n+2}^{(i)}} \\ \frac{\partial F_{2}^{(i)}}{\partial y_{n+\frac{1}{2}}^{(i)}} & \frac{\partial F_{2}^{(i)}}{\partial y_{n+1}^{(i)}} & \frac{\partial F_{2}^{(i)}}{\partial y_{n+\frac{3}{2}}^{(i)}} & \frac{\partial F_{2}^{(i)}}{\partial y_{n+2}^{(i)}} \end{bmatrix}^{-1} \begin{bmatrix} F_{\frac{1}{2}}^{(i)} \\ F_{1}^{(i)} \\ F_{\frac{3}{2}}^{(i)} \\ F_{2}^{(i)} \end{bmatrix} \quad (30)$$

Applying (27) in (30), therefore (30) becomes

$$\begin{bmatrix} \frac{\partial F_{\frac{1}{2}}^{(i)}}{\partial y_{n+\frac{1}{2}}^{(i)}} & \frac{\partial F_{\frac{1}{2}}^{(i)}}{\partial y_{n+1}^{(i)}} & \frac{\partial F_{\frac{1}{2}}^{(i)}}{\partial y_{n+\frac{3}{2}}^{(i)}} & \frac{\partial F_{\frac{1}{2}}^{(i)}}{\partial y_{n+2}^{(i)}} \\ \frac{\partial F_{1}^{(i)}}{\partial y_{n+\frac{1}{2}}^{(i)}} & \frac{\partial F_{1}^{(i)}}{\partial y_{n+1}^{(i)}} & \frac{\partial F_{1}^{(i)}}{\partial y_{n+\frac{3}{2}}^{(i)}} & \frac{\partial F_{1}^{(i)}}{\partial y_{n+2}^{(i)}} \\ \frac{\partial F_{\frac{3}{2}}^{(i)}}{\partial y_{n+\frac{1}{2}}^{(i)}} & \frac{\partial F_{\frac{3}{2}}^{(i)}}{\partial y_{n+1}^{(i)}} & \frac{\partial F_{\frac{3}{2}}^{(i)}}{\partial y_{n+\frac{3}{2}}^{(i)}} & \frac{\partial F_{\frac{3}{2}}^{(i)}}{\partial y_{n+2}^{(i)}} \\ \frac{\partial F_{2}^{(i)}}{\partial y_{n+\frac{1}{2}}^{(i)}} & \frac{\partial F_{2}^{(i)}}{\partial y_{n+1}^{(i)}} & \frac{\partial F_{2}^{(i)}}{\partial y_{n+\frac{3}{2}}^{(i)}} & \frac{\partial F_{2}^{(i)}}{\partial y_{n+2}^{(i)}} \end{bmatrix} \begin{bmatrix} e_{n+\frac{1}{2}}^{(i+1)} \\ e_{n+1}^{(i+1)} \\ e_{n+\frac{3}{2}}^{(i+1)} \\ e_{n+2}^{(i+1)} \end{bmatrix} = - \begin{bmatrix} F_{\frac{1}{2}}^{(i)} \\ F_{1}^{(i)} \\ F_{\frac{3}{2}}^{(i)} \\ F_{2}^{(i)} \end{bmatrix} \quad (31)$$

Equation (31) is equivalently written in matrix form as:

$$\underbrace{\begin{bmatrix} \left(1-\frac{3}{4(\rho+2)}h\frac{\partial F_{n+\frac{1}{2}}^{(i)}}{\partial y_{n+\frac{1}{2}}^{(i)}}\right) & 0 & 0 & 0 \\ -\frac{8}{3}\frac{\rho+8}{\rho+14} & \left(1-\frac{4}{\rho+14}h\frac{\partial F_{n+1}^{(i)}}{\partial y_{n+1}^{(i)}}\right) & 0 & 0 \\ \frac{15(4\rho-5)}{4\rho-61} & -\frac{45}{2}\frac{\rho-5}{4\rho-61} & \left(1+\frac{15}{4\rho-61}h\frac{\partial F_{n+\frac{3}{2}}^{(i)}}{\partial y_{n+\frac{3}{2}}^{(i)}}\right) & 0 \\ \left(-\frac{4(3\rho-16)}{\rho-54}-\frac{12\rho}{\rho-54}h\frac{\partial F_{n+2}^{(i)}}{\partial y_{n+\frac{1}{2}}^{(i)}}\right) & \frac{27(\rho-4)}{\rho-54} & -\frac{36}{5}\frac{\rho-16}{\rho-54} & \left(1-\frac{12}{\rho-54}h\frac{\partial F_{n+2}^{(i)}}{\partial y_{n+2}^{(i)}}\right) \end{bmatrix}}_{JacobianMatrix}$$

$$\begin{bmatrix} e^{(i+1)}_{n+\frac{1}{2}} \\ e^{(i+1)}_{n+1} \\ e^{(i+1)}_{n+\frac{3}{2}} \\ e^{(i+1)}_{n+2} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ \frac{8}{3}\frac{\rho+8}{\rho+14} & -1 & 0 & 0 \\ -\frac{15(4\rho-5)}{4\rho-61} & \frac{45}{2}\frac{\rho-5}{4\rho-61} & -1 & 0 \\ \frac{4(3\rho-16)}{\rho-54} & -\frac{27(\rho-4)}{\rho-54} & \frac{36}{5}\frac{\rho-16}{\rho-54} & -1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix} +$$

$$h\begin{bmatrix} 0 & -\frac{3\rho}{4(\rho+2)} & 0 & 0 \\ 0 & 0 & -\frac{4\rho}{\rho+14} & 0 \\ 0 & 0 & 0 & \frac{15}{4\rho-61} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{bmatrix} + h\begin{bmatrix} \frac{3}{4(\rho+2)} & 0 & 0 & 0 \\ 0 & \frac{4}{\rho+14} & 0 & 0 \\ 0 & 0 & -\frac{15}{4\rho-61} & 0 \\ \frac{12\rho}{\rho-54} & 0 & 0 & -\frac{12}{\rho-54} \end{bmatrix} \begin{bmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{bmatrix} +$$

$$\begin{bmatrix} \varepsilon_{\frac{1}{2}} \\ \varepsilon_1 \\ \varepsilon_{\frac{3}{2}} \\ \varepsilon_2 \end{bmatrix} \qquad (32)$$

A programme in C language is written to implement the equation (32).

Test Problems and Numerical Results

To validate the efficiency of the methods, the following systems of first order linear and non-linear stiff ordinary differential equations in initial value problems are solved

Problem 1

$$y_1' = 998y_1 + 1998y_2, \quad y_1(0) = 1, \qquad 0 \le x \le 20$$
$$y_2' = -999y_1 - 1999y_2, y_2(0) = 0,$$

Exact Solution:

$$y_1(x) = 2e^{-x} - e^{-1000x},$$
$$y_2(x) = -e^{-x} + e^{-1000x},$$

Eigenvalues: $\lambda = -1$ and $\lambda = -1000$.

Source:

Problem 2

$$y_1' = -100002y_1 + 100000y_2^2, y_1(0) = 1, 0 \le x \le 20$$
$$y_2' = y_1 - y_2(1 + y_2), y_2(0) = 0,$$

Exact Solution:

$$y_1(x) = 2e^{-x} - e^{-1000x},$$
$$y_2(x) = -e^{-x} + e^{-1000x},$$

Eigenvalues: $\lambda = -1$ and $\lambda = -100002$.

Problem 3

$$y_1' = 1195y_1 - 1995y_2, y_1(0) = 2, 0 \le x \le 20$$
$$y_2' = 1197y_1 - 1997y_2, y_2(0) = -2,$$

Exact Solution:

$$y_1(x) = 10e^{-2x} - 8e^{-800x},$$
$$y_2(x) = 6e^{-2x} - 8e^{-800x},$$

Eigenvalues: $\lambda = -2$and $\lambda = -800$
Source:
The numerical results are presented in Table 1-3 below. The initial value problems tested are solved with the proposed extended 2-point diagonally implicit super class of BBDF with off –step points when $\rho = -\frac{1}{2}$ and compared with the existing DI2BBDFO and DIE2SBBDFO in terms of Maximum error and Computation time. Therefore, the following notations are used in the tables below:
H: Total Number of Steps.
NS: Total Number of Steps.
$MAXE$: Maximum Error.
$TIME$: Computation Time (Second).
AHBBDF: Extended 2-point diagonally implicit super class of BBDF with off –step points when $= -\frac{1}{2}$
DI2BBDFO: Fifth order 2-point diagonally implicit BBDF with two off-step points method.
DIE2SBBDFO: Fifth order diagonally implicit extended 2-point super class of BBDF with two off-step points.

Table 1: Comparison of Maximum error and Computation Time for Problem 1

H	$METHOD$	NS	$MAXE$	$TIME$
10^{-2}	$D12BBDFO$	1000	$9.63369e + 002$	$2.29000e - 002$
	$DIE2OSBBDF$	1000	$3.64319e + 003$	$7.27300e - 002$
	$AHBBDF$	1000	$1.73416e + 098$	$9.60000e - 003$
10^{-3}	$D12BBDFO$	10000	$2.30943e - 002$	$1.27600e - 001$
	$DIE2OSBBDF$	10000	$2.33110e - 002$	$2.87100e - 001$
	$AHBBDF$	10000	$2.23842e - 002$	$2.56900e - 002$
10^{-4}	$D12BBDFO$	100000	$5.73377e - 003$	$1.11700e + 001$
	$DIE2OSBBDF$	100000	$5.91332e - 003$	$2.39500e + 001$
	$AHBBDF$	100000	$5.08539e - 003$	$2.43700e - 001$
10^{-5}	$D12BBDFO$	1000000	$7.58510e - 005$	$1.10400e + 001$
	$DIE2OSBBDF$	1000000	$8.33503e - 005$	$2.69500e + 001$
	$AHBBDF$	1000000	$6.67262e - 005$	$2.39400e - 001$
10^{-6}	$D12BBDFO$	10000000	$7.82952e - 007$	$1.13900e + 002$
	$DIE2OSBBDF$	10000000	$8.77479e - 007$	$2.55800e + 002$
	$AHBBDF$	10000000	$6.85450e - 007$	$2.53600e + 001$

Table 2: Comparison of Maximum error and Computation Time for Problem 2

H	$METHOD$	NS	$MAXE$	$TIME$
10^{-2}	$D12BBDFO$	1000	9.31522e + 011	2.83900e − 002
	$DIE2OSBBDF$	1000	1.18075e + 012	8.38800e − 002
	$AHBBDF$	1000	2.51767e + 013	6.23600e − 003
10^{-3}	$D12BBDFO$	10000	2.02250e + 000	1.73300e − 001
	$DIE2OSBBDF$	10000	5.50445e + 001	4.34600e − 001
	$AHBBDF$	10000	2.40834e + 007	3.41100e − 002
10^{-4}	$D12BBDFO$	100000	4.66074e − 007	1.41300e + 000
	$DIE2OSBBDF$	100000	1.80461e − 006	2.98200e + 000
	$AHBBDF$	100000	2.59835e + 007	3.04000e − 001
10^{-5}	$D12BBDFO$	1000000	1.92248e − 010	1.32700e + 001
	$DIE2OSBBDF$	1000000	2.00838e − 010	3.39500e + 001
	$AHBBDF$	1000000	1.62100e − 010	3.08200e + 000
10^{-6}	$D12BBDFO$	10000000	7.92305e − 011	1.52000e + 002
	$DIE2OSBBDF$	10000000	1.14193e − 010	3.30100e + 002
	$AHBBDF$	10000000	7.00794e − 011	3.34800e + 001

Table 3: Comparison of Maximum error and Computation Time for Problem 3

H	$METHOD$	NS	$MAXE$	$TIME$
10^{-2}	$D12BBDFO$	1000	1.62000e + 003	2.66400e − 002
	$DIE2OSBBDF$	1000	5.43597e + 003	4.60300e − 002
	$AHBBDF$	1000	9.98479e + 072	1.10900e − 002
10^{-3}	$D12BBDFO$	10000	2.63151e − 001	1.28100e − 001
	$DIE2OSBBDF$	10000	2.67252e − 001	2.57900e − 001
	$AHBBDF$	10000	2.49481e − 001	3.36800e − 002
10^{-4}	$D12BBDFO$	100000	3.12469e − 002	1.28200e + 000
	$DIE2OSBBDF$	100000	3.23524e − 002	2.89300e + 000
	$AHBBDF$	100000	2.76694e − 002	2.56800e − 001
10^{-5}	$D12BBDFO$	1000000	3.91104e − 004	1.05500e + 001
	$DIE2OSBBDF$	1000000	4.31195e − 004	2.65700e + 001
	$AHBBDF$	1000000	3.43686e − 004	2.51900e + 000
10^{-6}	$D12BBDFO$	10000000	4.01152e − 006	1.13000e + 002
	$DIE2OSBBDF$	10000000	4.49922e − 006	2.48700e + 002
	$AHBBDF$	10000000	3.51159e − 006	2.48300e + 001

The graphs of the scaled maximum error for the tested problems are given below, in order to give visual impact on the efficiency of the method, the efficiency curves of $\log_{10} MAXE$ against $\log_{10} h$ for the tested problems are plotted.

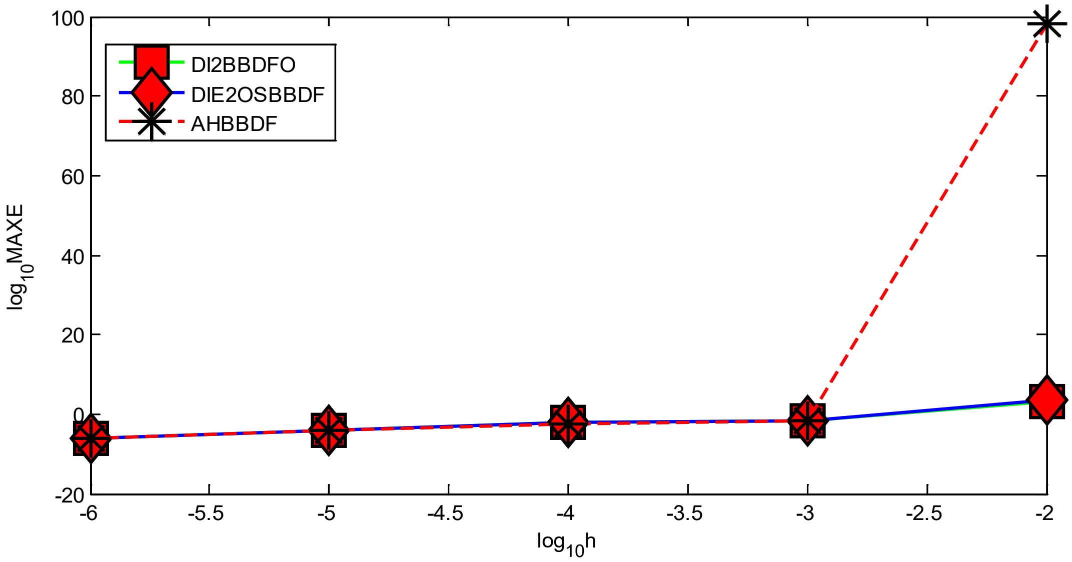

Figure 4: Efficiency Curve for problem 1

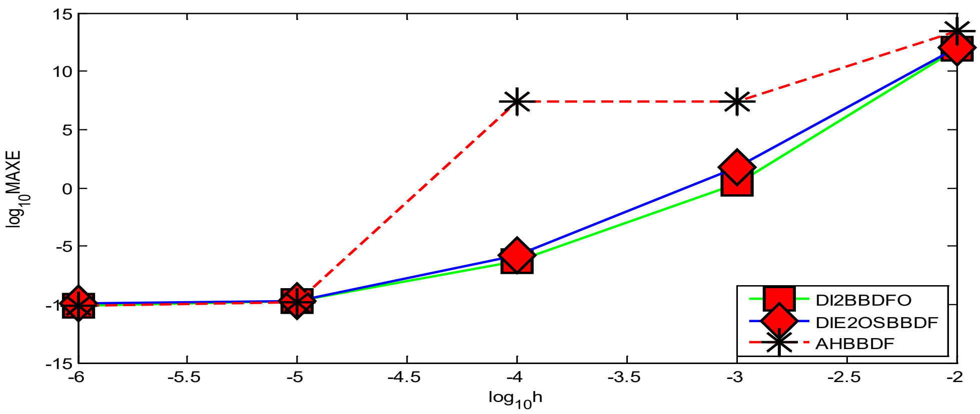

Figure 2: Efficiency Curve for problem 2

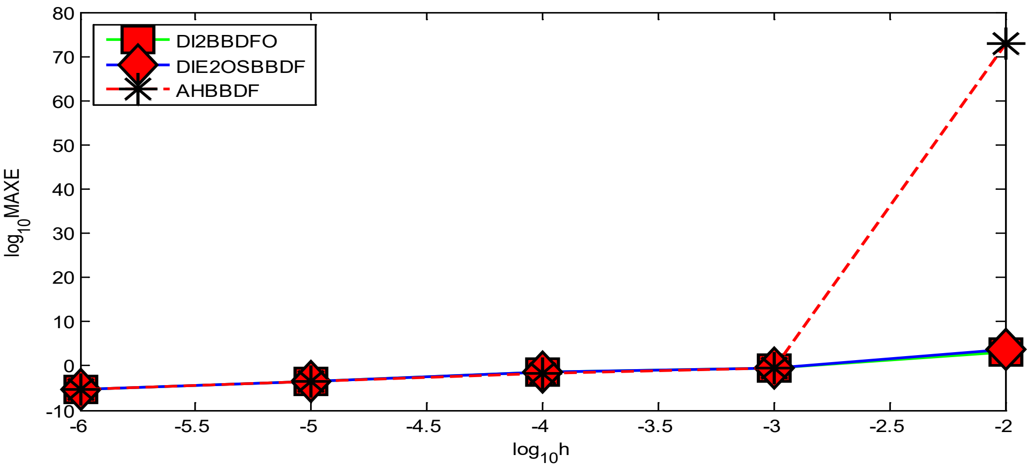

Figure 3: Efficiency Curve for problem 3

DISCUSSION

Tables 1 to 4 present a comparison of three different numerical results in terms of maximum error and computation time for three selected stiff problems (see problems 1 to 3). These results were obtained using the new method (i.e., AHBBDF for ρ = -1/2) and two existing methods: D12BBDFO for ρ = 0 and DIE2OBBDF for ρ = 1/5. From all three tested problems, we observe that the new method exhibits lower maximum error and shorter computation time. This makes it a more efficient method for solving systems of first-order stiff initial value problems (IVPs). The efficiency curves in Figures 1 to 3 further illustrate that the scaled maximum error for the new method is consistently lower than that of the two compared methods as we reduce the step size h. This serves as additional evidence that the new method converges faster than the other methods.

CONCLUSION

An A-stable hybrid block method for solving systems of first-order stiff initial value problems of ordinary differential equations (ODEs) has been developed. This method generates two different approximate solution values with two off-step points concurrently at each iteration step. By employing formula (2) in terms of a free parameter ρ, we have demonstrated that the method is of order 5 and converges. For absolute stability of the method, we selected different values of ρ within the interval -1 < ρ < 1 and demonstrated that the method is A-stable. Numerical results for some selected stiff initial value problems were obtained, and a comparison was made by applying the new method (i.e., AHBBDF) alongside existing D12BBDF and DIE2OBBDF methods. The AHBBDF method is observed to have a lower maximum error in all the tested problems, making it a more reliable and efficient solver for stiff initial value problems. The computation time is also competitive.

REFERENCES

Musa, H., Alhassan, B., and Abasi, N. (2022). Diagonally Implicit 2-point Block Backward Differentiation Formula with Two Off-Step Points for Solving Stiff Initial Value Problems.*Nigerian Journal of Mathematics and Applications*.**32** (**II**): 108-119.

Ibrahim, Z.B., Othman, K.I., and Suleiman, M.B. (2007). Implicit r-point Block Backward Differentiation Formula for first order stiff ODEs. *Applied Mathematics and Computational*, **186**:558-565.

Lambert, J.D. (1973). Computational methods in ODEs. New York: John Wiley.

Suleiman, M. B., Musa, H., Ismail, F., Senu, N., and Ibrahim, Z.B. (2014). A new super class of block backward differentiation formulas for stiff ODEs, Asian-European journal of mathematics.

Alhassan, B., and Musa, H. (2023). Diagonally Implicit Extended 2-Point Super Class of Block Backward Differentiation Formula with Two Off-step Points for Solving First Order Stiff Initial Value Problems. Applied Mathematics and Computational Intelligence Volume 12, No.1, PP. 101-124.

Abasi, N, Suleiman, M.B., Abbasi, N. and Musa, H. (2014). 2-point block BDF method with off-step points for solving stiff ODEs. Journal of Soft Computing and Applications 2014: 1-15.

Alt, R. (1978). A-stable one-step methods with step-size control for stiff systems of ordinary differential equations. J ComputAppl Math 4: 29-35.

Alvarez, J, and Rojo, J. (2002). An improved class of generalized Runge-Kuttamethods for stiff problems. Part I: The scalar case. Appl Math Comput 130: 537-560.

Cash, J.R. (1980). On the integration of stiff systems of ODEs using extended backward differentiation formulae. Numer Math 34: 235-246.

Yatim, S.A.M, Ibrahim, Z.B, Othman, K. I, and Suleiman, M.B. (2011). A quantitative comparison of numerical method for solving stiff ordinary differential equations. Math ProblEng 2011: 193691.

Musa, H, Suleiman, M. B, and Senu, N. (2012). A-stable 2-point block extended backward differentiation formulas for solving stiff ODEs. AIP Conference Proceedings 1450: 254-258.

Zawawi, I. S. M, Ibrahim, Z. B, and Othman, K.I. (2015). Derivation of diagonally implicit block backward differentiation formulas for solving stiff IVPs. Math ProblEng2015: 179231.